ECON 4910 Envvironmental economics; spring 2014

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Lecture note 6B: Optimal control theory

This note gives a brief, non-rigorous sketch of basic optimal control theory, which is a useful tool in several simple economic problems, such as those in resource economics and environmental economics.

Consider the dynamic optimization problem

$$\max \int_0^\infty e^{-rt} f(x(t), S(t), t) dt \tag{1}$$

subject to

$$\dot{S}(t) = g(x(t), S(t), t) \tag{2}$$

$$S(0) = S_0 \text{ historically given} \tag{3}$$

$$S(t) \geq 0 \text{ for all } t \tag{4}$$

where f and g are continuous and differentiable functions (and in many cases concave in (x, S)), and r is an exogenous positive discount rate. The variable $S(t)$ is a *stock* variable, also called a *state* variable, and can only change gradually over time as given by (2). The variable $x(t)$, on the other hand, is a variable that the decision maker chooses at any time. It is often called a *control* variable. In many economic problems the variable $x(t)$ will be constrained to be non-negative.

Remark 1: In the problem above there is only one control variable and one state variable. It is straightforward to generalize to many control and state variables, and the number of control variables need not be equal to the number of state variables.

Remark 2: The constraint (4) is more general than it might seem, as we often can reformulate the problem so we get this type of constraint. Assume e.g. that the constraint was $S(t) \leq \bar{S}$. We can then reformulate the problem by defining $Z(t) = \bar{S} - S(t)$, implying that $Z(t) \geq 0$. In this case the dynamic equation (2) must be replaced by $\dot{Z} = -g(x(t), \bar{S} - Z(t), t)$ and $S(t)$ in (1) must be replaced by $\bar{S} - Z(t)$.

The current value Hamiltonian

The current value Hamiltonian H is defined as

$$H(x, S, \lambda, t) = f(x, S, t) + \lambda g(x, S, t)$$

where $\lambda(t)$ is continuous and differentiable. The variable $\lambda(t)$ is often called a *co-state variable*. This variable will be non-negative in all problems where "more of the state variable" is "good". More precisely: The derivative of the maximized integral in (1) with respect to S_0 is equal to $\lambda(0)$. For this reason $\lambda(t)$ is also often called the *shadow price* of the state variable $S(t)$.

Conditions for an optimal solution

A solution to the problem (1)-(4) is a time path of the control variable $x(t)$ and an associated time path for the state variable $S(t)$. For optimal paths, there exist a differentiable function $\lambda(t)$ and a piecewise continuous function $\gamma(t)$ such that the following equations must hold for all t:

$$\frac{\partial H(x(t), S(t), \lambda(t), t)}{\partial x} = 0 \tag{5}$$

$$\dot{\lambda}(t) = r\lambda(t) - \frac{\partial H(x(t), S(t), \lambda(t), t)}{\partial S} - \gamma(t) \tag{6}$$

$$\gamma(t) \geq 0 \text{ and } \gamma(t)S(t) = 0 \tag{7}$$

$$Lim_{t \to \infty} e^{-rt} \lambda(t) S(t) = 0 \tag{8}$$

Remark 3: If $x(t)$ is constrained to be non-negative, (5) must be replaced by $\frac{\partial H}{\partial x} \leq 0$ and $\frac{\partial H}{\partial x} x(t) = 0$.

Remark 4: If we know from the problem that $S(t) > 0$ for all t, we can forget about $\gamma(t)$, since it always will be zero.

Remark 5: Condition (8) is a transversality condition. Transversality conditions are simple in problems with finite horizons, but considerably more complicated for problems with an infinite horizon (like our problem). The condition (8) holds for all problems where $\lambda(t) \geq 0$.

Remark 6: If f and g are concave in (x, S) and $\lambda(t) \geq 0$, the conditions (5)-(8) are sufficient for an optimal solution. If we can find a time path for $x(t)$ and for $S(t)$ satisfying (5)-(8) in this case, we thus know that the time paths $(x(t), S(t))$ are optimal.

Remark 7: As mentioned in Remark 1, it is straightforward to generalize to many control and state variables. If there are n state variables, there are also n co-state variables $(\lambda_1, ...\lambda_n)$, n Lagrangian multipliers $(\gamma_1...\gamma_n)$, and n differential equations of each of the types (2) and (6).